Extremal collision sequences of particles on a line: Optimal transmission of kinetic energy

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(Received 10 June 1999; revised manuscript received 2 February 2000; published 23 January 2001)

The transmission of kinetic energy through chains of inelastically colliding spheres is investigated for the case of constant coefficient of restitution $\epsilon=\text{const}$ and impact-velocity-dependent coefficient $\epsilon(v)$ for viscoelastic particles. We derive a theory for the optimal distribution of particle masses which maximize the energy transfer along the chain and check it numerically. We found that for $\epsilon=\text{const}$, the mass distribution is a monotonous function which does not depend on the value of ϵ. In contrast, for $\epsilon(v)$ the mass distribution reveals a pronounced maximum, depending on the particle properties and on the chain length. The system investigated demonstrates that even for small and simple systems, the velocity dependence of the coefficient of restitution may lead to new effects with respect to the same systems under the simplifying approximation $\epsilon=\text{const}$.

DOI: 10.1103/PhysRevE.63.021505 PACS number(s): 45.50.Tn, 61.85.+p

I. INTRODUCTION

Chains of nonlinear interacting particles have long been of large great interest, and a variety of interesting effects occurring in those systems has been described, such as solitons, (e.g., [1]), energy localization (e.g., [2]), etc. In the context of granular materials chains of inelastically colliding particles have been investigated as model systems for shaken granular material (e.g., [3,4]), granular compaction [5], and the ''inelastic collapse'' (e.g., [6,7]). The kinetic theory of one-dimensional granular systems has been addressed in [8].

In this paper, we consider a linear chain of inelastically colliding particles of masses m_i and radii R_i $(i=0\cdots n)$ with initial velocities $v_0=v>0$ and $v_i=0$ $(i=1\cdots n)$ at initial positions $x_i>x_j$ for $i>j$ with $x_{i+1}-x_i>R_{i+1}+R_i$ (Fig. 1). The masses of the first and last particles m_0 and m_n are given and we address the following question: How have the masses in between been chosen to maximize the energy transfer from the first particle of the chain to the last one? If n is a variable, how should n be chosen to maximize the after-collisional velocity v_n' of the last particle.

One can easily study the chains of ideally elastic spheres and of spheres interacting via a constant coefficient of restitution. It is much more complicated to deal with chains of viscoelastic particles, which have an impact-velocity-dependent coefficient and which, as we show below, exhibit quite unexpected behavior. It has been demonstrated recently that the kinetic properties of ''thermodynamically large'' systems of viscoelastic particles differ significantly from those of particles interacting with a constant coefficient of restitution [9]. The system considered in this paper may serve as an example of a *small* system whose properties change qualitatively when the viscoelastic properties of the particles are taken into account explicitly.

In the present study, the problem of the most efficient energy transmission in a chain of particles of variable mass is addressed. We analyze the optimal distribution for the particle masses and calculate the optimal size of the system.

II. ELASTIC PARTICLES

The textbook problem of elastic collisions may serve us to introduce the notation. Assume particle 0 collides with the resting particle 1. Then after the collision, the velocity of particle 1 is

$$v_1'=\frac{2m_0}{m_0+m_1}v_0 \tag{1}$$

(the primed variables refer to after-collisional velocities), and for a chain of $n+1$ particles of masses $m_0, \quad m_1,\dots,m_n$ one has analogously [10]

$$v_n'=2^n\prod_{k=0}^{n-1}\left(1+\frac{m_{k+1}}{m_k}\right)^{-1}v_0 . \tag{2}$$

For this system one finds easily that the choice $m_i=\sqrt{m_{i-1}m_{i+1}}$ $(i=2\cdots n-1)$ maximizes v_n'. If we fix m_0 and m_n, obviously the mass distribution

$$m_k=\left(\frac{m_n}{m_0}\right)^{k/n}m_0 \tag{3}$$

maximizes v_n':

$$v_n'=\left[\frac{2}{1+\left(\frac{m_n}{m_0}\right)^{1/n}}\right]^n v_0 . \tag{4}$$

The function $R_v=v_n'/v_0$ always increases with n and has the limit

$$R_v=\left[\frac{v_n'}{v_0}\right]_{n\to\infty}=\sqrt{\frac{m_0}{m_n}}, \tag{5}$$

i.e., if the masses of the particles are chosen according to Eq. (3), the kinetic energy of the first particle is completely transferred to the last one by a chain of infinite length.

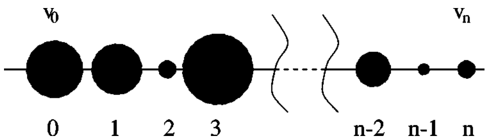

FIG. 1. Sketch.

For the case of dissipative collisions, an infinite chain cannot be optimal since in each collision energy is dissipated. Hence, we expect an optimum for the chain length for which the velocity of the last particle reaches its maximum.

III. PARTICLES WITH A CONSTANT RESTITUTION COEFFICIENT

According to our model, the particles collide pairwise. This allows us to use the restitution coefficient, which relates the relative velocity of colliding particles i and $i+1$ after collision to that before the collision:

$$\epsilon=\left|\frac{v'_{i+1}-v'_i}{v_{i+1}-v_i}\right|. \tag{6}$$

Equation (1) turns then into

$$v'_1=\frac{1+\epsilon}{1+(m_1/m_0)}v_0, \tag{7}$$

where we again assume that the particle with velocity v_0 and mass m_0 hits a particle of mass m_1 at rest, which starts moving with the velocity v'_1. Straightforward generalization of the previous analysis for the case of the dissipative collisions with a constant coefficient of restitution ϵ shows that the optimal mass distribution is identical to that for the elastic case (3). This means that the optimal mass distribution does not depend on the dissipation if $\epsilon=$const. The velocity of the last particle in the chain reads for this case

$$v'_n=\left[\frac{1+\epsilon}{1+\left(\frac{m_n}{m_0}\right)^{1/n}}\right]^n v_0. \tag{8}$$

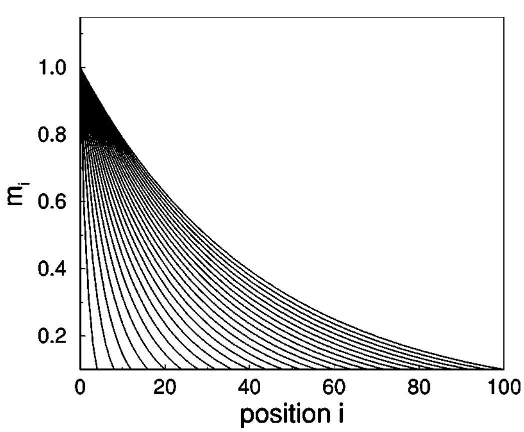

FIG. 2. Optimal mass distribution m_i, $i=1\cdots n$, for the case of a constant restitution coefficient ϵ. Each of the lines shows the mass m_i over the index i for a specified chain length n. The masses of the first and last particles are fixed at $m_0=1$ and $m_n=0.1$.

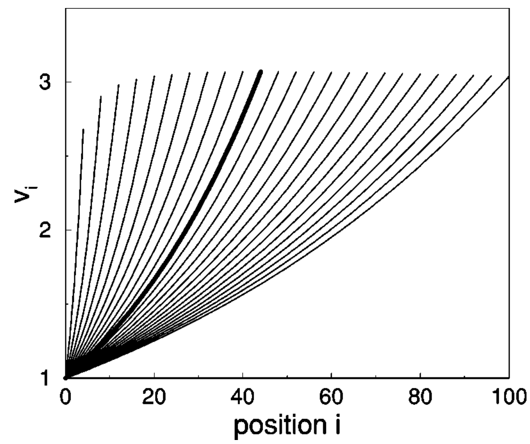

FIG. 3. Velocity distribution of particles in chains with the optimal mass distribution (given in Fig. 2) according to Eq. (3). Each of the lines shows the velocity v_i over the index i for a specified chain length n. The dissipative constant is $b=(1-\epsilon)=5\times10^{-4}$. The last particle reaches its maximal velocity for chain length $n^*=44$ (bold drawn). The velocity of the first particle of the chain is $v_0=1$.

Figure 2 shows the optimal mass distribution for different chain lengths n. The mass of the first particle is $m_0=1$ and of the last particle is $m_n=0.1$.

In the next section, we will consider particles which interact via a velocity-dependent coefficient of restitution. Since the velocity of the particles varies for the particles of the chain, we characterize the dissipation of the colliding spheres not by the coefficient of restitution itself but rather we define a dissipative constant b. For the case of a constant coefficient ϵ, it is defined as $b=(1-\epsilon)$.

In contrast to the mass distribution, the corresponding velocity distributions do depend on the value of the restitution coefficient ϵ. Figures 3 and 4 show the velocity distribution for two different values of the dissipative constant, $b=5\times10^{-4}$ and $b=0.032$.

For the case of dissipative collisions, the ratio $R_v=v'_n/v_0$ does not monotonously increase with n, but rather it has an extremum which shifts to smaller chain lengths with increasing dissipative parameter b. The optimal value of n, which maximizes R_v, reads

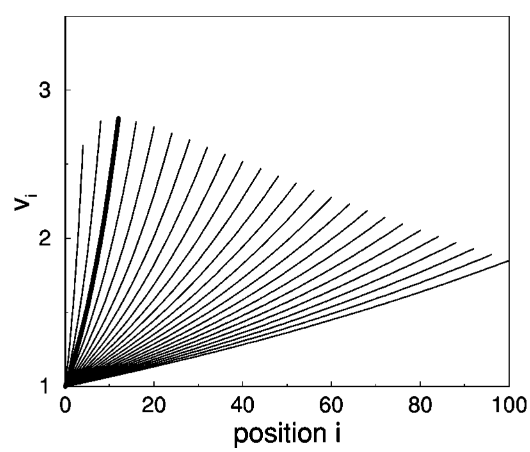

FIG. 4. Same as Fig. 3 but for $b=0.032$. The optimal chain length is $n^*=12$.

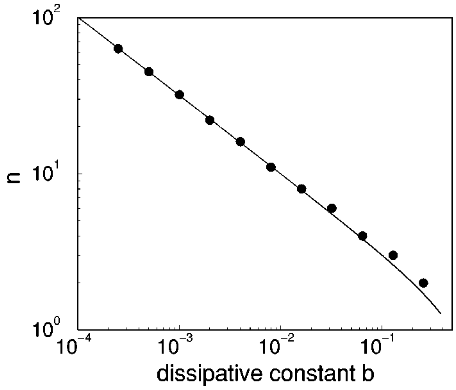

FIG. 5. The optimal chain length n^*, which gives the maximal transmission of energy along the chain with the fixed first and last masses, as a function of the dissipative parameter $b=(1-\epsilon)$. The line shows the prediction of Eq. (9), with x_0 found numerically. Points refer to the results of a direct numerical optimization of the masses in the chain.

$$n^*=\frac{\ln(m_n/m_0)}{\ln(x_0)}, \tag{9}$$

where x_0 is the solution of the equation

$$(1+x_0)=(1+\epsilon)x_0^{x_0/(1+x_0)}. \tag{10}$$

Correspondingly, the extremal value of the R_v reads

$$R_v^*=\left[\frac{1+\epsilon}{1+x_0}\right]^{n^*}. \tag{11}$$

In Fig. 5 the dependence of the extremal n^* on the restitution coefficient is shown.

IV. VISCOELASTIC PARTICLES

A. Collisional law for the viscoelastic particles

It has been shown that for colliding viscoelastic spheres, the restitution coefficient depends on the masses of the colliding particles and also on their relative velocity v_{ij} [11]. An explicit expression for the coefficient of restitution is given by the series [12,13]

$$\epsilon=1-C_1\left(\frac{3A}{2}\right)\alpha^{2/5}v_{ij}^{1/5}+C_2\left(\frac{3A}{2}\right)^2\alpha^{4/5}v_{ij}^{2/5}\mp\cdots \tag{12}$$

with

$$\alpha=\frac{2Y\sqrt{R^{\mathrm{eff}}}}{3m^{\mathrm{eff}}(1-\nu^2)}, \tag{13}$$

where Y is the Young modulus and ν is the Poisson ratio. The effective mass and effective radius are defined as $R^{\mathrm{eff}}=R_iR_j/(R_i+R_j)$ and $m^{\mathrm{eff}}=m_im_j/(m_i+m_j)$, where $R_{i/j}$ and $m_{i/j}$ are radii and masses of the colliding particles. The constant A describing the dissipative properties of the spheres depends on material parameters (for details, see [11]). The

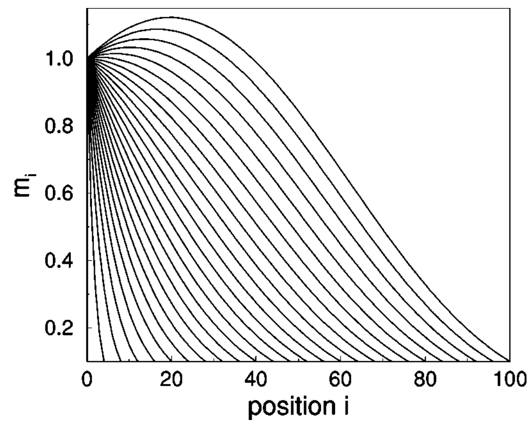

FIG. 6. Optimal mass distribution m_i, $i=1\cdots n$, for the case of viscoelastic particles with the restitution coefficient given by Eq. (14) with $b=5\times10^{-4}$. Each of the lines shows the mass m_i over the index i for a specified chain length n. The masses of the first and last particles are $m_0=1$ and $m_n=0.1$.

constants $C_1=1.153\,44$ and $C_2=0.798\,26$ were obtained analytically in Ref. [12] and then confirmed by numerical simulations.

For the following calculation we neglect terms $O(v^{2/5})$ and of higher orders. Moreover, we also assume for simplicity that all particles are of the same radius R, but have different masses [14]. We abbreviate

$$\epsilon=1-b\frac{v_{ij}^{1/5}}{(m^{\mathrm{eff}})^{2/5}} \tag{14}$$

with

$$b=C_1\left(\frac{3A}{2}\right)\left(\frac{2}{3}\frac{Y\sqrt{R/2}}{1-\nu^2}\right)^{2/5}. \tag{15}$$

Thus, the collision with $\epsilon=$const and given dissipative constant b, as introduced above, corresponds (i.e., has equal value of ϵ) to the viscoelastic collision with the same b, with unit effective mass $m^{\mathrm{eff}}=1$ and unit relative velocity v_{ij}.

Hence, for viscoelastic particles the velocity of the $k+1$st particle after colliding with the kth particle reads

$$v'_{k+1}=\frac{2-b\left(\frac{m_{k+1}+m_k}{m_{k+1}m_k}\right)^{2/5}v_k^{1/5}}{1+\frac{m_{k+1}}{m_k}}v_k. \tag{16}$$

The masses m_k, $k=1\cdots n-1$, which maximize v'_n can be determined numerically and the results are shown in Figs. 6 and 7 for two different values of the dissipative constant b.

For small chain length or small b, respectively, the optimal mass distribution is very close to that for the elastic chain as shown in Fig. 2. Again, we find a monotonously decaying function for the masses. For larger chain length n or larger dissipation b, however, the mass distribution is a nonmonotonous function. The according velocities of the particles in chains of spheres of optimal masses are drawn in

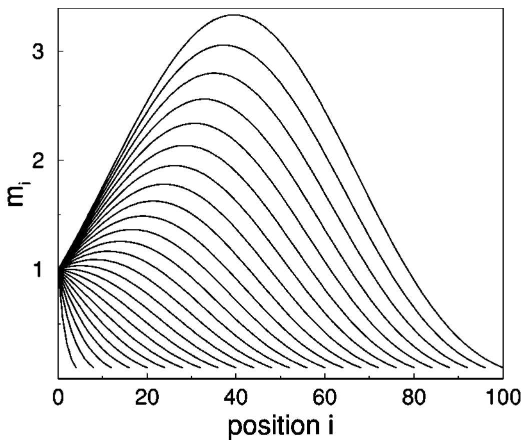

FIG. 7. The same plot as Fig. 6 but for $b=2\times10^{-3}$.

Figs. 8 and 9. Note that the mass distribution and velocity distribution are related by Eq. (16).

B. Variational approach to the optimal mass distribution

In the following, we describe an approximative theory of the optimal collision chain of viscoelastic particles. To this end we first evaluate the loss of kinetic energy in the chain, which we divide into two parts and term as ''inertial'' and ''viscous'' losses. In our approach, we treat the part of the energy which is not transformed from the first particle of the chain to the last one as a ''lost'' energy. In this sense, the energy is ''lost'' according to two mechanisms. The first is due to a mismatch of subsequent masses, which causes incomplete transfer of momentum even for an elastic collision when the masses differ (this part of the energy loss is called ''inertial''). The second refers to the dissipative nature of collisions and, therefore, this loss is called ''viscous'' below. The inertial loss in the collision, attributed to the energy transfer to the ith particle, is thus given by the energy which remains in the $(i-1)$st particle after the collision:

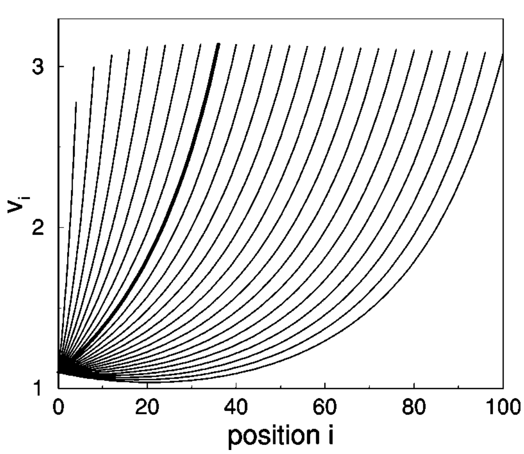

FIG. 8. Velocity distribution for viscoelastic particles in chains with the optimal mass distribution given in Fig. 6. Each of the lines shows the velocity v_i over the index i for a specified chain length n. The dissipative constant is $b=5\times10^{-4}$. The last particle reaches its maximal velocity for the chain length $n^*=36$ (bold drawn). The velocity of the first particle of the chain is $v_0=1$.

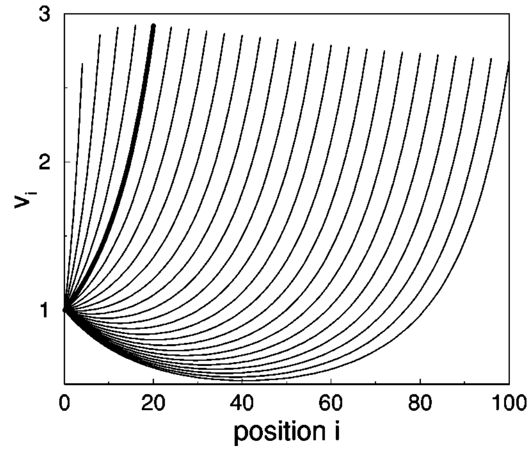

FIG. 9. The same plot as Fig. 8 but for the mass distribution according to Fig. 7 ($b=2\times10^{-3}$). The optimal chain length is $n^*=20$.

$$\Delta E_{in}^{(i)}=\frac{m_{i-1}}{2}(v'_{i-1})^2=\frac{m_{i-1}}{2}\left(\frac{m_i-m_{i-1}}{m_i+m_{i-1}}\right)^2 v_{i-1}^2. \quad (17)$$

For long enough chains, we approximate the discrete mass distribution by a continuous one, $m(x)$. This, with the assumption of small mass gradients, gives $m_i\approx m_{i-1}+[dm(x)/dx]1$, where we assume that particles are separated on a line by a unit distance. Within the continuum picture $\Delta E_{\rm in}^{(i)}\to(dE_{\rm in}/dx)1$, and we write for the ''line density'' of the inertial loss, discarding high-order mass gradients,

$$\frac{dE_{\rm in}}{dx}\approx\frac{\left(\frac{dm(x)}{dx}\right)^2}{8m(x)}v(x)^2. \quad (18)$$

Viscous losses describe the energy losses according to the inelastic properties of the material, therefore they are equal to the difference of the kinetic energy of a particle after an *elastic* collision (with no dissipation) and that after a *dissipative* collision,

$$\begin{aligned}\Delta E_{\rm vis}^{(i)}&=\frac{m_i v_i^2}{2}\bigg|_{\epsilon=1}-\frac{m_i v_i^2}{2}\bigg|_{\epsilon=\epsilon(v_i)}\\&=\frac{m_i}{2}\left(\frac{2}{1+\frac{m_i}{m_{i-1}}}\right)^2 v_{i-1}^2-\frac{m_i}{2}\left(\frac{1+\epsilon(v_{i-1})}{1+\frac{m_i}{m_{i-1}}}\right)^2 v_{i-1}^2\\&=\frac{2m_i v_{i-1}^2}{\left(1+\frac{m_i}{m_{i-1}}\right)^2}\left\{1-\left[1-\frac{b}{2}\left(\frac{m_i+m_{i-1}}{m_i m_{i-1}}\right)^{2/5}v_{i-1}^{1/5}\right]^2\right\}.\end{aligned} \quad (19)$$

Now we assume that the dissipative parameter b is small, so that one can keep only the linear term, expanding $\Delta E_{\rm vis}^{(i)}$ with respect to b. Transforming then to continuous variables and discarding terms which are products of b and mass gradients (which are also supposed to be small) yields

$$\frac{dE_{\rm vis}}{dx} \approx \frac{b}{2^{3/5}} m^{3/5} v^{11/5}. \tag{20}$$

Thus, the total energy loss in the entire chain reads

$$E_{\rm tot} = \int_0^n \left[\frac{m_x^2}{8m} v^2 + \frac{b}{2^{3/5}} m^{3/5} v^{11/5} \right] dx, \tag{21}$$

where $m_x \equiv dm/dx$. As it follows from Eq. (21), to evaluate $E_{\rm tot}$ one needs the velocity distribution $v(x)$. As a zero-order approximation we use an ''ideal chain ansatz.'' This refers to a velocity distribution $v(x)$ in an idealized chain, where the kinetic energy completely transforms through the chain, i.e., where $\frac{1}{2} m(x) v^2(x) = \text{const} = \frac{1}{2} m_0 v_0^2$. With $m_0 = 1$, $v_0 = 1$, so that $v(x) = 1/\sqrt{m(x)}$, this ansatz yields

$$E_{\rm tot} = \int_0^n \left[\frac{m_x^2}{8m^2} + \frac{b}{2^{3/5}} \frac{1}{m^{1/2}} \right] dx. \tag{22}$$

The mass distribution which minimizes $E_{\rm tot}$ satisfies the Euler equation applied to the integrand in Eq. (22):

$$\frac{d}{dx} \frac{2m_x}{8m^2} - \frac{\partial}{\partial m} \left[\frac{m_x^2}{8m^2} + \frac{b}{2^{3/5}} \frac{1}{m^{1/2}} \right] = 0. \tag{23}$$

Equation (23) leads to an equation for the mass distribution of the optimal chain, written for $y(x) \equiv 1/m(x)$:

$$\frac{d^2 y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - 2^{2/5} b y^{3/2} = 0. \tag{24}$$

Multiplying Eq. (24) by $2(y'/y^2)$ (y is always positive), we recast Eq. (24) into the form

$$\frac{d}{dx} [(y'/y)^2 - 4 \times 2^{2/5} b y^{1/2}] = 0 \tag{25}$$

which implies the first integral of this equation:

$$(y'/y)^2 - 4 \times 2^{2/5} b y^{1/2} = -c, \tag{26}$$

where the constant c depends on parameter b, the chain length n, and initial and final masses, m_0 and m_n. The form of the solution depends on the sign of this constant. If the mass distribution has an extremum at $x = x^*$, such that $m'(x^*) = 0$ and $y'(x^*) = 0$, the constant c is positive. This follows from Eq. (26), i.e., $c = 4 \times 2^{2/5} b y^{1/2}(x^*) > 0$, since $y^{1/2}(x^*)$ is positive.

The solution of the *first-order* equation (26) may be found straightforwardly. The general solution is somewhat lengthy, but for the case of $m_0 = 1$ (one can always use the appropriate mass unit), this reads (for $c > 0$)

$$y(x) = m(x)^{-1} = \frac{c^2}{2^{4/5} b^2} \cos^{-4} \left(\frac{x\sqrt{c}}{2} + \varphi \right), \tag{27}$$

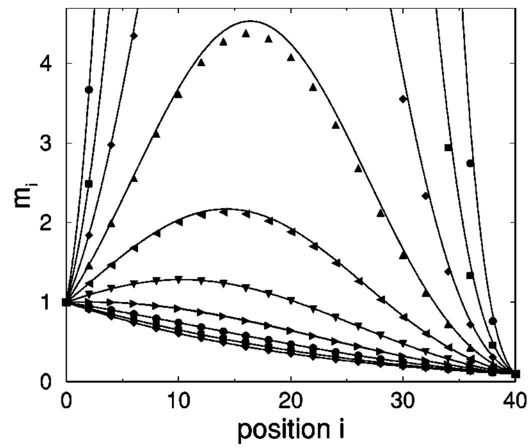

FIG. 10. Mass distribution in chains of viscoelastic particles of length $n = 40$ with optimal mass distribution for different values of the dissipative parameter b. Lines, results of the variational theory, according to Eq. (24); points, numerical optimization (from top to bottom: ●, $b = 0.128$; ■, $b = 0.064$; ◆, $b = 0.032$; ▲, $b = 0.016$; ◀, $b = 0.008$; ▼, $b = 0.004$; ▶, $b = 0.002$; etc.). As previously, m_i is the mass of the ith particle along the chain.

where

$$\cos \varphi = \sqrt{\frac{c}{2^{2/5} b}}. \tag{28}$$

The value of the constant c may be found from the second boundary condition $y(n) = 1/m_n$, which yields a transcendental equation for c:

$$\cos\left(\frac{n\sqrt{c}}{2} \right) - \sin\left(n \frac{\sqrt{c}}{2} \right) \sqrt{\frac{2^{2/5} b}{c} - 1} = \frac{2^{2/5} b}{c} m_n^{-1/4}. \tag{29}$$

The last equation has to be solved numerically. Instead, however, we solved numerically directly the initial differential Eq. (24).

Note that some scaling properties of the solution may be deduced just from the form of Eq. (24). Namely, as it follows from this equation, the solution should depend on the reduced length variable $x\sqrt{b}$. Thus, the distribution of masses for chains with different chain length n and different dissipative constant b should coincide after rescaling the particle numbers as $i \to \sqrt{b} i$, provided the masses m_0 and m_n are the same for these chains. We will consider the scaling properties of the mass distribution in more detail later.

Figure 10 shows the optimal mass distribution for a chain of length $n = 40$ for different damping parameters b. The lines display the (numerical) solution of the variation Eq. (24), whereas the points show the results of a numerical optimization of the chain problem. For small dissipation b, both results agree.

For larger values of b, the solution of the variational equation (24) deviates from the results of the numerical optimization. This follows from the fact that for larger b, the gradients of the mass distribution are not small and our variational approach loses its accuracy. Note, however, that

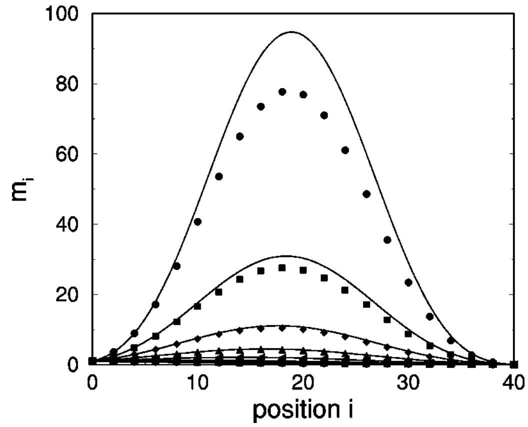

FIG. 11. Same data and symbols as in Fig. 10 but plotted in a larger scale.

while the absolute values of the masses in the mass distribution deviate from that given by the variational approach, this still predicts well the position of the maximum of the distribution. Figure 11 shows the same data as Fig. 10 but for larger dissipation parameter b.

Figure 12 displays the velocity distribution for the optimal chain with the mass distribution shown in Fig. 11. The data given in Fig. 12 refer to the numerical optimization, where Eq. (16), which relates velocity and mass distribution, is used. According to the maximum in the mass distribution, the velocity distribution reveals for larger b a pronounced minimum.

One can give a simple physical explanation of the appearance of a maximum in the mass distribution (and correspondingly a minimum in the velocity distribution): As it is seen from Eq. (14), the restitution coefficient increases with decreasing impact velocity and increasing masses of colliding particles this reduces the viscous losses. Thus slowing down particles, by increasing their masses in the inner part of the

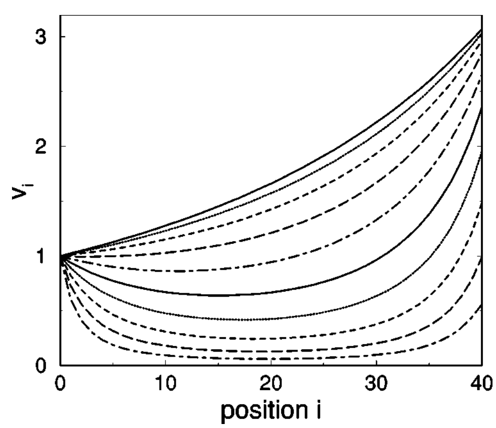

FIG. 12. The velocity distribution in chains of viscoelastic particles of length $n=40$ with the optimal mass distribution according to Fig. 11 for different values of the dissipative constant b. Lines from top to bottom: $b=2.5\times 10^{-4}$, 5×10^{-4}, 0.001, 0.002, 0.004, 0.008, 0.016, 0.032, 0.064, and 0.128. The velocity distribution is obtained from the mass distribution (given in Fig. 11) according to Eq. (16). As previously, v_i is the velocity of the ith particle along the chain.

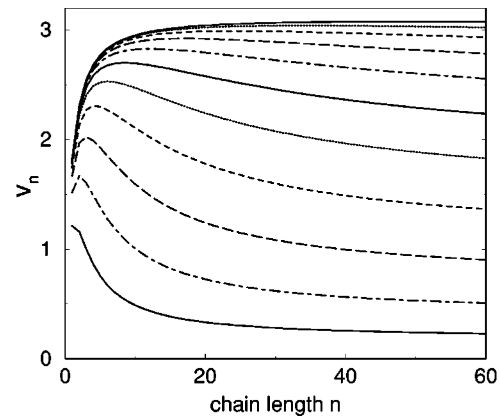

FIG. 13. Velocity of the last particle v_n for chains of viscoelastic particles with optimal mass distribution over the chain length n for different values of b. As in Fig. 12, the velocity distribution was obtained from the mass distribution according to Eq. (16), and lines from top to bottom correspond to $b=2.5\times 10^{-4}$, 5×10^{-4}, 0.001, 0.002, 0.004, 0.008, 0.016, 0.032, 0.064, and 0.128. Note that with increasing dissipative constant b the maximum of $v_n(n)$, which corresponds to the optimal chain length n^*, shifts to smaller values of n, which means naturally that optimal chains are shorter for larger dissipation.

chain leads to a decrease of the viscous losses of the energy transfer. The larger the masses in the middle and the smaller their velocities, the less energy is lost due to dissipation. On the other hand, since masses m_0 and m_n are fixed, very large masses in the middle of the chain will cause a large mass mismatch of the subsequent masses and thus large inertial losses [see Eq. (17)]. The optimal mass distribution minimizing the *total* losses compromises (dictated by b) between these two opposite tendencies. For the case of a constant coefficient of restitution, the relative part of the kinetic energy, which is lost due to dissipation, does not depend on the impact velocity. This means that only minimization of the inertial losses, caused by mass gradient, may play a role in the optimization of the mass distribution. Thus only a monotonous mass distribution with minimal mass gradients along the chain may be observed as an optimal one for the case of the constant restitution coefficient.

As in the case of the constant restitution coefficient, the velocity of the last particle v_n' of an *optimal* chain depends on n. For short chains (with m_0, m_n fixed), the mass gradient of adjacent particles is large, hence inertia losses are large as well. For very long chains, viscous losses become large. Hence, we expect that among the optimal chains there exists a chain with a certain length n^* which allows for an optimal transmission of kinetic energy from the first particle to the last one. Figure 13 shows the velocity of the last particle for chains with optimal mass distribution as a function of the chain length n for different values of the dissipative parameter b. Naturally, as for the case of the constant restitution coefficient, the optimal chain length n^* shifts to smaller values with increasing dissipative constant b.

Having the mass distribution and the velocity distribution obtained from the numerical optimization one can check directly the validity of the ‘‘ideal chain ansatz,’’ $v(x)=1/\sqrt{m(x)}$, used in the variational approach. In Fig. 14, we

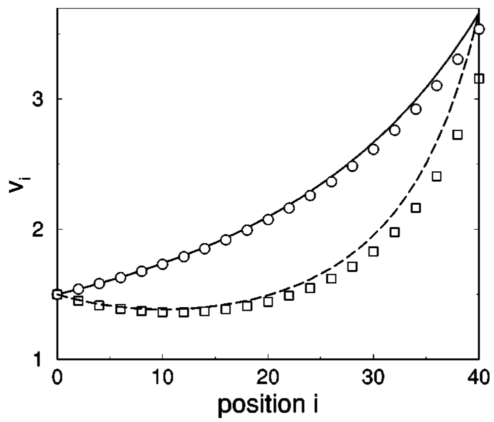

FIG. 14. The velocity distribution in chains of viscoelastic particles of length $n=40$ with the optimal mass distribution according to Fig. 10. Lines give the velocity distribution for the ideal chain ansatz, $v_i=1/\sqrt{m_i}$ (with masses taken from the optimization data); points show the numerical optimization data for $b=0.001$ (top) and $b=0.008$ (bottom). Note that for these values of the dissipative parameter b, the variational theory gives a very accurate description for the optimal mass distribution (see Fig. 10).

compare $v(x)$ obtained by optimization with that from the ansatz. As it is seen from the figure, the ideal chain ansatz turns out to be rather accurate for small dissipation parameter b and for the initial part of the chain. It demonstrates, however, noticeable deviations from the optimization data for larger b, especially at the end of the chain, i.e., for $i\approx n$. This is not surprising since it uses an assumption of complete transmission of energy, which is definitely poor for the very end of the chain. On the other hand, as it follows from Figs. 10 and 11, this ansatz yields rather accurate results when applied to the mass distribution problem. The possible explanation for this follows from the boundary condition for the mass distribution at the end of the chain, $m(i=n)=m_n$. This imposes the correct behavior of the mass distribution at this part of the chain and partly compensates for the inaccuracy of the velocity distribution, which develops mainly at the chain end (see Fig. 14).

C. Scaling laws for the optimal mass distribution

Now we analyze how the maximal mass $m^*\equiv m(x^*)$ (the mass of the heaviest sphere located at $x=x^*$) in the optimal mass distribution depends on the chain length n and the dissipative parameter b. We show that there exists a simple scaling relation between these values.

We start from Eq. (27) for the optimal mass distribution,

$$m(x)=\frac{2^{4/5}b^2}{c^2}\cos^4\left(\frac{x\sqrt{c}}{2}+\varphi\right), \tag{30}$$

with c and φ defined by Eqs. (28) and (29). The condition for the optimal mass

$$m_x(x^*)=-\frac{2^{9/5}b^2}{c^{3/2}}\cos^3\left(\frac{x^*\sqrt{c}}{2}+\varphi\right)\sin\left(\frac{x^*\sqrt{c}}{2}+\varphi\right)=0 \tag{31}$$

implies $\sin[(x^*\sqrt{c}/2)+\varphi]=0$ and thus the relation between the maximal mass m^* and the constant c,

$$m^*=\frac{2^{4/5}b^2}{c^2}\cos^4\left(\frac{x^*\sqrt{c}}{2}+\varphi\right)=\frac{2^{4/5}b^2}{c^2}, \tag{32}$$

i.e.,

$$c=2^{2/5}b/\sqrt{m^*}. \tag{33}$$

This allows us to write the boundary condition for $m(x)$ at $x=n$:

$$m_n=m^*\cos^4\left(\frac{n\sqrt{c}}{2}+\varphi\right) \tag{34}$$

or equivalently

$$\frac{n\sqrt{c}}{2}=\arccos\left[\left(\frac{m_n}{m^*}\right)^{1/4}\right]-\varphi. \tag{35}$$

Simple analysis shows that $\varphi<0$ if the optimal distribution has a maximum [this follows from the form of the solution (30) and the requirement that $m(x)$ increases at $x=0$]. Thus, one obtains from Eqs. (28) and (33)

$$\varphi=-\arccos\left[\left(\frac{m_0}{m^*}\right)^{1/4}\right]. \tag{36}$$

Using again Eq. (33) for the constant c, we recast Eq. (35) into the final form

$$n\sqrt{b}=2^{4/5}(m^*)^{1/4}\left\{\arccos\left[\left(\frac{m_n}{m^*}\right)^{1/4}\right]+\arccos\left[\left(\frac{m_0}{m^*}\right)^{1/4}\right]\right\}. \tag{37}$$

This scaling relation expresses the product $n\sqrt{b}$ in terms of the maximal mass m^*. For the case of a strongly pronounced maximum in the optimal mass distribution, i.e., when $m_0/m^*\ll 1$ and $m_n/m^*\ll 1$, one can expand the arccos(x) in Eq. (37) to obtain a linear scaling relation between $(m^*)^{1/4}$ and $n\sqrt{b}$:

$$n\sqrt{b}=p(m^*)^{1/4}-q, \tag{38}$$

with

$$p=2^{4/5}\pi, \tag{39}$$

$$q=2^{4/5}(m_0^{1/4}+m_n^{1/4}). \tag{40}$$

In Fig. 15 we compare the analytical relation (37) and its linear approximation (38) with the results for m^*, following from the numerical optimization for the mass distribution for different chain lengths and different dissipative constants. As one can see from Fig. 15, the results of the analytical theory and of the numerical optimization agree well, except for large dissipation values. We would like to stress that there are no fitting parameters used.

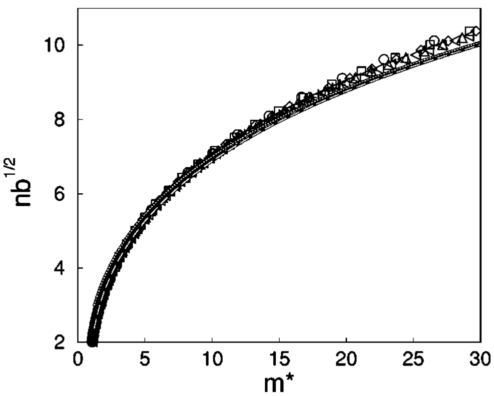

FIG. 15. $n\sqrt{b}$ as a function of m^* for the chain of viscoelastic particles with the optimal mass distribution. Here m^* is the mass of the heaviest particle in the chain, n is a chain length, and b is the dissipative parameter. In the figure we plotted $n\sqrt{b}$ over m^* for about 3000 different combinations of b and n ($n=2\cdots 300$, $b=0.0001\cdots 0.256$) including all data presented in Figs. 6, 7, 10, and 11. Without any adjustable parameters, the data from the numerical optimization of chains agree well with the analytical expressions Eq. (37), given by the dashed line. The linear approximation for the scaling relation, Eq. (38), is shown by the dotted line.

Using the optimal mass distribution, Eq. (30), one can compute the total energy loss in the chain, as given by Eq. (22):

$$E_{\rm tot}=\frac{nc}{2}+2\sqrt{c}\left\{\frac{\sqrt{1-\cos^2\varphi_n}}{\cos\varphi_n}-\frac{\sqrt{1-\cos^2\varphi}}{\cos\varphi}\right\}+2\sqrt{c}\{\arcsin[\cos\varphi_n]-\arcsin[\cos\varphi]\}, \tag{41}$$

where $\varphi_n=n\sqrt{c}/2+\varphi$. According to Eq. (30), one obtains

$$m_0=\frac{2^{4/5}b^2}{c^2}\cos^4\varphi, \tag{42}$$

$$m_n=\frac{2^{4/5}b^2}{c^2}\cos^4\varphi_n\,, \tag{43}$$

which allows us to express all trigonometric functions in Eq. (41) in terms of m_0 and m_n, yielding

$$E_{\rm tot}=2\left\{\sqrt{\frac{2^{2/5}b}{m_n^{1/2}}-c}-\sqrt{\frac{2^{2/5}b}{m_0^{1/2}}-c}\right\}-\frac{cn}{2}, \tag{44}$$

and finally, taking into account Eq. (33) for c, we arrive at the relation for the total losses

$$E_{\rm tot}(n,b)=2^{6/5}\sqrt{b}\left\{\sqrt{\frac{1}{\sqrt{m_n}}-\frac{1}{\sqrt{m^*}}}-\sqrt{\frac{1}{\sqrt{m_0}}-\frac{1}{\sqrt{m^*}}}-\frac{n\sqrt{b}}{2^{9/5}\sqrt{m^*}}\right\}. \tag{45}$$

Using the approximation for the maximal mass,

$$m^*\approx(n\sqrt{b}/p+q/p)^4, \tag{46}$$

which follows from Eq. (38), one obtains an explicit *approximate* relation for the total losses and, thus, for the final velocity

$$v'^2_n=\frac{m_0v_0^2}{m_n}-\frac{2}{m_n}E_{\rm tot}(n,b) \tag{47}$$

in terms of the chain length and the dissipation constant b. Unfortunately, due to the fact that chains with optimal lengths obviously do not have a maximum in their mass distribution, one cannot use the previous relations to estimate the optimal chain length for a given dissipation constant b, since these relations hold true only for chains which do have a maximum.

Note that since the maximal mass m^* depends only on the product $n\sqrt{b}$, the expression in curled brackets on the right-hand side of Eq. (45) also depends only on this combination. This suggests the following scaling relations for the final velocity for the chains with fixed $n\sqrt{b}$:

$$v'^2_n=m_n^{-1}-d\sqrt{b},$$
$$v'^2_n=m_n^{-1}-d'/n, \tag{48}$$

where we take into account that $m_0=1$, $v_0=1$, and where d and d' are some constants which are defined by the particular value of $n\sqrt{b}$.

V. CONCLUSION

We investigated analytically and numerically the transmission of kinetic energy through one-dimensional chains of inelastically colliding spheres, where the first and the last mass are fixed. For the case of a constant coefficient of restitution, we found that in the chain with optimal energy transmission, the mass of each particle is given by the geometric average of its neighbors, i.e., the distribution of the masses of the spheres is a monotonous, exponentially decreasing function. This function is independent of the coefficient of restitution ϵ, where the special case of elastically colliding particles ($\epsilon=1$) is included. We derived an expression for the chain length n^* which leads for a given ϵ to the optimal energy transfer (provided the masses in between the first and last mass have been chosen properly).

The situation changes qualitatively if we assume that the chain consists of viscoelastic spheres for which the coefficient of restitution depends on the impact velocity. Here, the optimal mass distribution which leads to maximum energy transfer is not necessarily a monotonous function. Depending on the chain length n and on the material parameters of the spheres, it may reveal a pronounced maximum. We consider the part of the kinetic energy of the first particle, which has not been transfered to the last one, as losses of energy. These losses have been characterized as losses according to incomplete transfer of momentum due to mass mismatch of the particles (inertia losses) and losses due to the dissipative nature of particle collisions (viscous losses). We develop a theory which describes the total energy losses along the chain, so that the optimal mass distribution, minimizing the losses, may be obtained as a solution of a variational equation. We find a general solution to this nonlinear second-order differential equation. Implication of the boundary conditions yields, however, a transcendental equation, which one needs to solve numerically (in practice, we solve numerically the initial differential equation). We observed that our variational theory agrees well with the results of the numerical optimization for the mass distribution, provided the dissipative material parameter is not too large. We also performed a direct verification of the basic approximation used in our variational approach.

From the exact solution of the variational equation, we obtained an analytical expression which relates the heaviest mass in the mass distribution to the chain length and the dissipation constant. We found that this analytical expression, having no fitting parameters, is in good agreement with the numerical data. Using the exact solution for the optimal mass distribution, we also found an expression for the total energy losses. This allowed us to obtain scaling relations which show how the velocity of the last particle in the chain scales with the length of the chain n and with the dissipation constant b, for the chain with the value of $n\sqrt{b}$ fixed.

It has been demonstrated before that for the case of ''thermodynamically large'' granular systems the impact-velocity dependence of the restitution coefficient, as it is given for viscoelastic particles, may lead to qualitatively different behavior as compared to systems with a constant restitution coefficient, e.g., [3,9,15]. The system investigated here may serve as an example of the major influence of the velocity dependence of the restitution coefficient even for relatively small (''lab scale'') and simple systems. Therefore, in general, the assumption of a constant coefficient of restitution is an approximation whose justification cannot be assumed *a priori* but has to be checked for each particular application.

ACKNOWLEDGMENT

The authors thank Thomas Schwager for valuable discussions.

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$$\epsilon = 1 - 2^{-1/5} b \left(\frac{4}{\pi\rho}\right)^{1/15} \left(\frac{m_i^{1/3} m_j^{1/3}}{m_i^{1/3} + m_j^{1/3}}\right)^{1/5} \left(\frac{m_i + m_j}{m_i m_j}\right)^{1/5} v_{ij}^{1/5}.$$

Numerical calculations show that the assumption of equal particle density does not change the results qualitatively, however the analysis will be much more complicated if not impossible.

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